INVITED PAPER

Nonparametric Bootstrap Test in a Multivariate Spatial-Temporal Model: A Simulation Study

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The assumptions of constant characteristics across spatial locations and constant characteristics across time points facilitates estimation in a multivariate spatial-temporal model. A test based on the nonparametric bootstrap in proposed to verify these assumptions. The simulation studies confirm that the proposed test procedures are powerful and correctly sized.

Keywords: coverage probability, robustness, spatial-temporal model

1. Introduction

A phenomenon that varies over space and time is sometimes viewed only from the spatial viewpoint or over the temporal perspective. The essence of spatial analysis is that "space matters" and what happens in one region is related to events in neighboring regions. This has been properly stated in what Tobler (1979) refers to as the First Law of Geography that states: "Everything is related to everything else, but closer things more so." In the statistical science, this is aptly captured in the concept of spatial autocorrelation.

Spatial dependence refers to the correlation of an attribute measured at two locations. In the absence of spatial dependence, the distance of the two locations does not influence the joint behavior of attributes observed at those two locations. When spatial dependence is present (for example, positive correlation), then nearby observations are more similar than those far apart. The analysis can be enhanced by drawing on temporal dependencies, in addition to spatial dependence. Failure to account for time dependencies means lost information. Temporal dependence means that events at one time can be influenced by what has happened in the past alone, whereas, spatial dependence implies that events at any one point in time can be influenced by both the past and the future (Anselin and Bera, 1998).

Landagan and Barrios (2007) proposed a spatial-temporal model that accommodates irregularly shaped spatial units, with temporal observations made at equal intervals of time shown in Equation 1.

$$Y_{it} = X_{it}\beta + W_{it}\gamma + \varepsilon_{it}, \;\; i = 1, \ldots, n \quad t = 1,2,\ldots, T \tag{1}$$

where Y_{it} is the response variable from locations i at time t, X_{it} is the set of covariates from location i and at time t, w_{it} is the set of variables in the neighborhood system of location i at time t, and ε_{it} are the error components. The error component was examined for dependence structures, but only the autoregressive behavior was explored as $\varepsilon_{it} = \mu_i + v_i$, where $\mu_i \sim IID(0,\sigma_\mu^2)$ and the remainder disturbances v_{it} following stationary AR(p). An estimation procedure that imbeds the Cochrane-Orcutt procedure into the backfitting algorithm of Hastie and Tibshirani (1990) for additive models was modified to simultaneously estimate group parameters at some point of the iterative process. Barrios and Landagan (2007) assumed: (i) constant covariate (β) effect across locations and time, (ii) constant temporal effect (ρ) across locations, and (iii) constant spatial effect (γ) across time.

Guarte and Barrios (2013) proposed a nonparametric bootstrap inference procedure based onthe Landagan and Barrios (2007) spatial-temporal model to verify assumptions (ii) and (iii). On the other hand, Martines (2008) proposed a procedure to estimate parameters of multivariate spatial-temporal model by imbedding a multivariate regression and vector autoregressive (VAR) model in backfitting algorithm. The multivariate spatial model at a minimum, subsumed an autoregressive error process adopted from the work of Singh et. al. (2005) given by the model:

$$\boldsymbol{Y}_i = \boldsymbol{X}_i\boldsymbol{\beta} + \boldsymbol{u}_i, \;\; \boldsymbol{u}_i = \boldsymbol{W}_i\rho + \boldsymbol{v}_i \tag{2}$$

where $\boldsymbol{Y}_i$ is the 1×r vector of observation on the response variable from i$^{\text{th}}$ location; $\boldsymbol{X}$ is the vector of covariates from i$^{\text{th}}$ location; $\boldsymbol{\beta}$ is the vector of parameters; and u_i is the vector of error component; $\boldsymbol{W}_i$ is the neighborhood variable; $\boldsymbol{\rho}$ is the corresponding spatial effect on $\boldsymbol{u}_i$; and $\boldsymbol{v}_i$ is the remainder disturbance which is distributed with mean zero and constant variance. Martines (2008), in order to capture the temporal effect, examined the error disturbance for dependence structure, but he only explored the VAR behavior and postulated it as $\boldsymbol{v}_i=\gamma v_{t-1}+\boldsymbol{\eta}_t$. The multivariate spatial-temporal model of Martines (2008), assuming additivity is given in Equation 3.

$$\boldsymbol{Y}_t = \boldsymbol{X}_t\boldsymbol{\beta} + \boldsymbol{W}_t\rho + \boldsymbol{v}_{t-1}\gamma + \boldsymbol{\eta}_t \tag{3}$$

Following Landagan and Barrios (2007), Martines (2008) assumed constant temporal effect across locations, and constant spatial effect across time periods in model (3).

This study focuses on multivariate spatial-temporal model and proposes a nonparametric bootstrap test in verifying the assumptions of model (3) by extending the work of Guarte and Barrios (2013) to the case of multivariate spatial-temporal model. Recognizing the theoretical importance of determinants, in this paper, the determinant estimates of the bivariate VAR(1) coefficients matrix of either multivariate characteristics across spatial locations/time points were obtained for the purpose of representing and preserving the parameter estimates and in characterizing the behavior of a multivariate spatial-temporal model.

2. Methodology

Given the spatial-temporal VAR model:

$$\underset{\sim}{y}_i(t) = \underset{\sim}{\phi}_{1i}\underset{\sim}{y}_i(t-1) + \underset{\sim}{\phi}_{2i}\underset{\sim}{y}_i(t-2) + \ldots + \underset{\sim}{\phi}_{pi}\underset{\sim}{y}_i(t-p) + \underset{\sim}{\varepsilon}_i(t), \tag{4}$$

in the same context as Martines (2008) assume (i) constant multivariate characteristics across spatial locations and (ii) constant multivariate characteristics across time points.

To verify these assumptions, the bivariate VAR(1) series characterized by the parameter $\underset{\sim}{\phi}_{1i} = \begin{bmatrix} \phi_{11}^i & \phi_{12}^i \\ \phi_{21}^i & \phi_{22}^i \end{bmatrix}$ with innovations from normal distribution with mean $\underset{\sim}{0}$ and variance $\underset{\sim}{\Sigma}_{\varepsilon_i(t)}$ were used. The bivariate VAR(1) series are characterized by,

$$\underset{\sim}{y}_i(t) + \underset{\sim}{\phi}_{1i}\underset{\sim}{y}_i(t-1) + \underset{\sim}{\varepsilon}_i(t) \tag{5}$$

Considering the spatial units, the bivariate VAR(1) series are available for N locations each with T time points. The following hypotheses are tested:

H_0: $\underset{\sim}{\phi}_{11} = \underset{\sim}{\phi}_{12} = \ldots = \underset{\sim}{\phi}_{1N}$, i.e., all spatial locations have the same multivariate characteristics over time.

H_1: $\underset{\sim}{\phi}_{1i} \neq \underset{\sim}{\phi}_{1j}$ for at least one pair of $i \neq j$ i.e., at least one spatial location differs multivariate characteristics over time.

Algorithm 1:

Given these N time series each with 2 dimensions, the following procedures are used in testing the multivariate characteristics across spatial locations:

1. For each location i estimate VAR(1) process on a bivariate specification and conditioning on $\underset{\sim}{y}_i(t-1)$, the empirical distribution of the centered residuals is,

$$\varepsilon_i(t) \sim WN(\underset{\sim}{0}, MSE). \quad (6)$$

2. Using the residuals (6), generate k bootstrap samples for each spatial location i of sample size $n\left(\underset{\sim}{\varepsilon}_i(1), \underset{\sim}{\varepsilon}_i(2), \ldots, \underset{\sim}{\varepsilon}_i(n)\right)$.
3. For every bootstrap sample in Step 2, generate k time series for every i^{th} spatial location using the estimated model in Step 1.
4. Estimate the bivariate VAR(1) model for every simulated time series in Step 3, then take the determinant of these bivariate VAR(1) coefficients matrix. Thus, there will be k determinants for k bootstrap samples.
5. Compute the standard error of the determinant of the coefficients $\left|\widehat{\underset{\sim}{\phi}_{1i}}\right|$ using the corresponding k bootstrap determinants $\left|\widehat{\underset{\sim}{\phi}^*_{1ij}}\right|$ of the multivariate characteristics parameter estimates $\hat{\underset{\sim}{\phi}}^*_{1ij}$ generated in Step 4. Then we have,

$$\hat{\sigma}_{\left|\widehat{\underset{\sim}{\phi}_{1i}}\right|} = \left[\frac{1}{k-1}\sum_{j=1}^{k}\left(\left|\widehat{\underset{\sim}{\phi}^*_{1ij}}\right| - \overline{\left|\widehat{\underset{\sim}{\phi}^*_{1ij}}\right|}\right)^2\right] \qquad \overline{\left|\widehat{\underset{\sim}{\phi}^*_{1ij}}\right|} = \frac{1}{k}\sum_{j=1}^{k}\left|\widehat{\underset{\sim}{\phi}^*_{1ij}}\right|, \quad (7)$$

where $j = 1, 2, \ldots$, k (bootstrap samples) and * = bootstrap estimates,

$\left|\widehat{\underset{\sim}{\phi}^*_{1ij}}\right|$ - determinant of the j bootstrapped estimates, in this case the estimated 2×2 square matrix of the multivariate characteristics across spatial locations, for j = 1, 2, …, k bootstraps.

$\overline{\left|\widehat{\underset{\sim}{\phi}^*_{1ij}}\right|}$ - mean of the determinant of the j bootstrapped estimates, in this case the estimated 2×2 square matrix of the multivariate characteristics across spatial locations, for j = 1, 2,…, k bootstraps.

6. Construct (1 – α) x 100%bootstrap confidence interval on each determinant of multivariate characteristics $\left|\underset{\sim}{\phi}_{1i}\right|$ estimated in Step 1:

$$\left|\widehat{\underset{\sim}{\phi}_{1i}}\right| \mp y_{\frac{\alpha}{2}}\hat{\sigma}_{\left|\widehat{\underset{\sim}{\phi}_{1i}}\right|} \quad (8)$$

where $\left|\widehat{\underset{\sim}{\phi}_{1i}}\right|$ is the determinant of the original point estimate from Step 1, and $\hat{\sigma}_{\left|\widehat{\underset{\sim}{\phi}_{1i}}\right|}$ is its standard error estimate from Step 5.

7. Compute the mean/median of the determinant of the multivariate characteristic parameter estimate $\left|\widehat{\underset{\sim}{\phi}_{1i}}\right|$ in Step 1.
8. Reject the null hypothesis that there is constant multivariate characteristic across spatial locations with (1−α)×100% coverage probability if more

than α% of the constructed intervals fail to contain the mean/median value computed in Step 7.

Using VAR(1) process of the bivariate specification, the data were transformed into cross-sectional for testing the constant multivariate characteristic across time points. The multivariate linear regression model was assumed to be the appropriate model for the cross-sectional data. By extending the univariate model of Guarte and Barrios (2013), the values of $\underset{\sim}{y}_i(t)$ were first translated in location and

$$x_{it} = \left(\frac{\underset{\sim}{y}_i'(t) - \underset{\sim}{\phi}_{1i}\underset{\sim}{y}_i'(t-1)}{\underset{\sim}{\beta}_{1t}} \right). \tag{9}$$

The above equation was derived from the following relationships:

$$\underset{\sim}{y}_i(t) = \underset{\sim}{\phi}_{1i}\underset{\sim}{y}_i(t-1) + \underset{\sim}{\varepsilon}_i(t), \quad \underset{\sim}{\varepsilon}_i(t) \sim N(\underset{\sim}{0}, \underset{\sim}{I}_2)$$

$$\underset{\sim}{y}_i'(t) = \underset{\sim}{y}(t)\underset{\sim}{I}_2\sigma + \underset{\sim}{u}_{it}, \quad \underset{\sim}{u}_{it} = \begin{bmatrix}100 & 100\end{bmatrix}, \quad \begin{matrix} i = 1,2,\dots,N \\ t = 2,3,\dots,T \end{matrix}, \quad \sigma{=}10$$

$$\underset{\sim}{y}_i'(t) = \underset{\sim}{\phi}_{1i}\left(\underset{\sim}{y}_i(t-1)\underset{\sim}{I}_2\sigma + \underset{\sim}{u}_{it}\right) + \underset{\sim}{\varepsilon}_i(t)$$

$$\underset{\sim}{y}_i'(t) = \underset{\sim}{\phi}_{1i}\underset{\sim}{y}_i'(t-1) + \underset{\sim}{x}_{it}\underset{\sim}{\beta}_{1t}$$

Now, using the covariates (9), the response variable was simulated using the regression model below:

$$\underset{\sim}{y}_{it} = \underset{\sim}{x}_{it}\underset{\sim}{\beta}_{1t} + \underset{\sim}{\varepsilon}_{it}, \quad \underset{\sim}{\varepsilon}_{it} \sim N(\underset{\sim}{0}, \underset{\sim}{I}_2\sigma^2), \quad \begin{matrix} i = 1,2,\dots,N \\ t = 2,3,\dots,T \end{matrix}, \tag{10}$$

For this study, $\underset{\sim}{\underset{\sim}{\beta}}_{1t} = \begin{bmatrix} \beta_{11}^t & \beta_{12}^t \\ \beta_{21}^t & \beta_{22}^t \end{bmatrix}$ for both the response $\underset{\sim}{y}_{it}$ and the predictor $\underset{\sim}{x}_{it}$. Given the series, we test the following hypotheses:

H_0: $\underset{\sim}{\beta}_{11} = \underset{\sim}{\beta}_{12} = \dots = \underset{\sim}{\beta}_{1T}$, i.e., multivariate regression coefficients have similar characteristics over time, across spatial locations.

H_1: $\underset{\sim}{\beta}_{1i} \neq \underset{\sim}{\beta}_{1j}$ for at least one pair of $i \neq j$ i.e., the regression coefficient vary in at least one pair of over time points across spatial locations.

Algorithm 2:

Model (10) is used in the following steps for test constant multivariate characteristics across time points:

1. Estimate the coefficient of model (10), and take the determinant of this, $\left|\widehat{\underset{\sim}{\beta}_{1t}}\right|$, for each time point.

2. Generate *k* bootstrap samples from the pairs of the dependent and independent variables for each time points,

$$b_j = \begin{pmatrix} \underset{\sim}{\boldsymbol{x}}_{it} \\ \underset{\sim}{\boldsymbol{y}}_{it} \end{pmatrix}, \tag{11}$$

$i = j$, implies $k = N$ bootstrap samples. The regression bootstrap procedure above is called case resampling.

3. Estimate the coefficients of model (10) again, and take the determinant, $\left|\widehat{\underset{\sim}{\beta}^{*}_{1tj}}\right|$, using the bootstrap samples obtained in the preceding step for each time point.

4. Compute the standard error of the estimated coefficients $\hat{\underset{\sim}{\beta}}_{1t}$ from Step 1 using the estimated bootstrap determinants, $\left|\widehat{\underset{\sim}{\beta}^{*}_{1tj}}\right|$ from Step 3. Then,

$$\hat{\sigma}_{\left|\widehat{\underset{\sim}{\beta}1t}\right|} = \left[\frac{1}{k-1} \sum_{j=1}^{k} \left(\left|\widehat{\underset{\sim}{\beta}^{*}_{1tj}}\right| - \overline{\left|\widehat{\underset{\sim}{\beta}^{*}_{1tj}}\right|} \right)^{2} \right] \quad \overline{\left|\widehat{\underset{\sim}{\beta}^{*}_{1tj}}\right|} = \frac{1}{k} \sum_{j=1}^{k} \left|\widehat{\underset{\sim}{\beta}^{*}_{1tj}}\right|, \tag{12}$$

where $j = 1, 2, \ldots, k$ (bootstrap samples) and * = bootstrap estimates,

$\left|\widehat{}_{tj}\right|$ - determinant of the j bootstrapped estimates, in this case the estimated 2×2 square matrix of the multivariate characteristics across time points, for j = 1, 2, …, k bootstraps.

$\overline{\left|\widehat{\underset{\sim}{\beta}^{*}_{1tj}}\right|}$ - mean of the determinant of the j bootstrapped estimates, in this case the estimated 2×2 square matrix of the multivariate characteristics across time points, for j = 1, 2, …, k bootstraps.

5. Construct (1 – α) x 100% bootstrap confidence interval on each determinant of multivariate characteristics $\left|\underset{\sim}{\beta}_{1t}\right|$ estimated in Step 1:

$$\left|\widehat{\underset{\sim}{\beta}_{1t}}\right| \mp y_{\frac{\alpha}{2}} \hat{\sigma}_{\left|\widehat{\underset{\sim}{\beta}1t}\right|}, \tag{13}$$

where $\left|\widehat{\underset{\sim}{\beta}_{1t}}\right|$ is the determinant of the original point estimate from Step 1, and $\hat{\sigma}_{\left|\widehat{\underset{\sim}{\beta}_{1t}}\right|}$ is its standard error estimate from Step 5.

6. Compute the mean/median of the determinant of the multivariate characteristic parameter estimate $\left|\widehat{\underset{\sim}{\beta}}_{1t}\right|$ in Step 1.
7. Reject the null hypothesis that there is constant multivariate characteristic across time points with $(1-\alpha)\times 100\%$ coverage probability if more than $\alpha\%$ of the constructed intervals fail to contain the mean/median value computed in Step 6.

Algorithm 3:

Power of the test is computed after Step 7 of algorithm 1 and after Step 6of algorithm 2. The lower and upper limits of the 95% confidence interval were used to compute their corresponding y-scores. That is, for a given confidence interval of the $\left|\widehat{\underset{\sim}{\phi}}{}^{*}_{1ij}\right|$, (LL,UL). The y-score of the LL (lower limit) is,

$$y_{LL} = \frac{LL - \overline{\left|\widehat{\underset{\sim}{\phi}}_{1i}\right|}}{\hat{\sigma}_{\left|\widehat{\phi}_{1i}\right|}} \quad \text{and/or} \quad y_{LL} = \frac{LL - \widetilde{\left|\widehat{\underset{\sim}{\phi}}_{1i}\right|}}{\hat{\sigma}_{\left|\widehat{\phi}_{1i}\right|}}, \tag{14}$$

$$y_{LL} = \frac{LL - \overline{\left|\widehat{\underset{\sim}{\beta}}_{1t}\right|}}{\hat{\sigma}_{\left|\widehat{\beta}_{1t}\right|}} \quad \text{and/or} \quad y_{LL} = \frac{LL - \widetilde{\left|\widehat{\underset{\sim}{\beta}}_{1t}\right|}}{\hat{\sigma}_{\left|\widehat{\beta}_{1t}\right|}}, \tag{15}$$

where $\overline{\left|\widehat{\underset{\sim}{\phi}}_{1i}\right|}$ and $\widetilde{\left|\widehat{\underset{\sim}{\phi}}_{1i}\right|}$ are the mean and the median value, respectively, of the determinant of the coefficients from the bootstrap samples for multivariate characteristic across spatial locations, and $\overline{\left|\widehat{\underset{\sim}{\beta}}_{1t}\right|}$ and $\widetilde{\left|\widehat{\underset{\sim}{\beta}}_{1t}\right|}$ are the mean and the median values, respectively, of the determinant of the coefficients from the bootstrap samples for multivariate characteristics across time points. And the y-scores for the UL (upper limit),

$$y_{UL} = \frac{UL - \overline{\left|\widehat{\underset{\sim}{\phi}}_{1i}\right|}}{\hat{\sigma}_{\left|\widehat{\phi}_{1i}\right|}} \quad \text{and/or} \quad y_{UL} = \frac{UL - \widetilde{\left|\widehat{\underset{\sim}{\phi}}_{1i}\right|}}{\hat{\sigma}_{\left|\widehat{\phi}_{1i}\right|}}, \tag{16}$$

$$y_{UL} = \frac{UL - \left|\overline{\widehat{\underset{\sim}{\beta}}_{1t}}\right|}{\hat{\sigma}_{\left|\widehat{\beta}_{1t}\right|}} \quad \text{and/or} \quad y_{UL} = \frac{UL - \left|\widetilde{\widehat{\underset{\sim}{\beta}}_{1t}}\right|}{\hat{\sigma}_{\left|\widehat{\beta}_{1t}\right|}} \tag{17}$$

Then β is,

$$\beta = P(y_{UL} \leq X \leq y_{LL}). \tag{18}$$

And thus the power of the test is,

$$\text{Power} = 1 - P(y_{UL} \leq X \leq y_{LL})$$
$$= 1 - [P(y \leq y_{UL}) - P(y \leq y_{LL}) \quad (19)$$

3. Simulation Studies

Four dataset were simulated, of which, two are from balanced datasets N=T=100 and N=T=20 and the other two sets are from unbalanced datasets N=70, T=50 and N=40, T=35.

To evaluate the test of constant multivariate characteristics across spatial locations, consider the stationary VAR(1) with simplest bivariate specification model given below:

$$\boldsymbol{y}_t = \begin{pmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{pmatrix} y_{t-1} + \varepsilon_t \quad with \;\; \Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}. \quad (20)$$

Algorithms 1 and 3 were utilized for the needed results. The simulation process has no intercept term for the model.

The same algorithms were applied also to the case of nonconstant multivariate characteristics across spatial locations. Four datasets were generated using (20) model with the first five spatial locations having the same multivariate characteristics for each dataset. In each dataset generated, there were four sets of results and each has either 6 or 7 cases. The first set has a coefficient, Q_{11} that changes with an increment of 0.10 while holding all other coefficients unchanged, the second set, with a coefficient, Q_{12} that changes with also the same increment of 0.10 while all other coefficients were held fixed. The third set with coefficient Q_{21} likewise changes with an increment of 0.10 and the last, with coefficient of Q_{22} also changes with similar increment of 0.10.

The same stationary bivariate VAR(1) mentioned above was used to evaluate the test of constant multivariate characteristics across time points. The datasets were then transformed into cross-sectional and the multivariate linear regression model was assumed to be the appropriate model for the cross-sectional data. Covariates were computed from equation (5) using $\beta_1 = \begin{pmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{pmatrix}$. The response variable was then computed using these covariates and parameter β_1. Algorithms 2 and 3 were applied to obtain the necessary results.

In the case of the nonconstant multivariate characteristics across time points, four datasets were generated with the first 5% time points having the same multivariate characteristics. In each dataset generated, there were four sets of results and each has either 6 or 7 cases. The first set has a coefficient, β_{11}that changes with an increment of 0.10. The second set, with a coefficient, β_{12} changes with also the same increment of 0.10. The third set with coefficient β_{21} changes with an increment of 0.10, and the last, with coefficient of β_{22} also changes with similar increment of 0.10.

4. Results

From Table 1, of the 100 bootstrap normal confidence intervals that were constructed, only four (4) failed to contain both the mean and median. The four (4)spatial locations that significantly differed from the rest were the third highest multivariate characteristics estimate (0.584), the highest estimate (0.615), the lowest estimate (0.224), and the second highest estimate (0.599). This could mean that even if the population of spatial locations was homogeneous with respect to the multivariate characteristics, sampling variation will induce some spatial locations to be different in the sample with respect to multivariate characteristics. We can conclude that the testing procedure was able to correctly identify the true situation and is properly sized for this dataset. Not rejecting the null hypothesis of constant multivariate characteristics across spatial locations with 95% coverage probability actually captured the fact that not more than five (5) spatial locations differ in multivariate characteristics. There was a correct inference for this large balanced data, $(N, T) = (100,100)$.

Table 1. Multivariate Characteristics Across Spatial Locations for Stationary Bivariate VAR(1) Model for Large and Small Balanced/Unbalanced Data and 95% Coverage Probability

Sample Size	Criterion	Value	95% Coverage Probability
(N,T) = (100,100)	Mean	0.414	Do not reject H_0 (4)
	Median	0.407	Do not reject H_0 (4)
(N,T) = (70,50)	Mean	0.377	Do not reject H_0 (2)
	Median	0.386	Do not reject H_0 (3)
(N,T) = (40,35)	Mean	0.344	Do not reject H_0 (1)
	Median	0.340	Do not reject H_0 (1)
(N,T) = (20,20)	Mean	0.320	Do not reject H_0 (0)
	Median	0.348	Do not reject H_0 (0)

Note: Figures in parentheses are the number of normal bootstrap confidence intervals that failed to contain the mean/or median.

For large unbalanced dataset, $N=70$, $T=50$, the null hypothesis of constant multivariate characteristics across 70 spatial locations based on both the mean and/or median is not rejected with 95% coverage probability using the bootstrap normal confidence intervals. Of the 70 bootstrap confidence intervals constructed, only two (2) failed to contain the mean and three (3) failed to contain the median. The three (3) spatial locations that significantly differed from the sixty seven (67) based on the median were the highest estimate (0.591), the second lowest estimate

(0.169) and the lowest estimate (0.093). Again, due to sampling variation, this leads to a few spatial locations coming out different with respect to multivariate characteristics in the said sample. We may therefore conclude that the test procedure was able to correctly identify the true situation and is properly sized for this dataset using bootstrap normal confidence intervals. This means not rejecting the null hypothesis of constant multivariate characteristics across spatial locations with 95% coverage probability actually captured the fact that not more than four (4) spatial locations differ in multivariate characteristics effect.

For small unbalanced dataset, N=40, T=35, the null hypothesis of constant multivariate characteristics effect across spatial locations was not rejected with 95% coverage probability. Of the 40 bootstrap confidence intervals constructed, only one (1) failed to contain both the mean and median estimates. Thus, we can conclude that the test procedure was able to correctly identify the true situation and is properly sized for this dataset. That is, not rejecting the null hypothesis of constant multivariate characteristics across spatial locations with 95% coverage probability actually captured the fact that not more than two (2) spatial locations differ in multivariate characteristics.

For small balanced dataset, N=20, T=20, the null hypothesis of constant multivariate characteristics across spatial locations was not rejected with 95% coverage probability. Of the 20 confidence intervals constructed, none failed to contain both the mean and median estimates using the bootstrap normal confidence intervals. We may therefore conclude that the testing procedure was able to correctly identify the true situation and is properly sized for this dataset. That is, not rejecting the null hypothesis of constant multivariate characteristics across spatial locations with 95% coverage probability actually captured the fact that not more than one (1) spatial location differ in multivariate characteristics, which in this case, there was none.

The power plots in Figure 1, show the estimated maximum power with 5% different multivariate characteristics across spatial locations for large balanced dataset. The horizontal axis represents the parameter value (Q_{ii}) with an increment of 0.10 and the vertical axis represents the estimated maximum power based on either the mean (denoted by black line) or the median (denoted by red line). The estimated maximum power of the test procedure is high on the diagonal of VAR(1) coefficient matrix (set 1_1 and set 1_4) compared to that of the off-diagonal (set 1_2) and set 1_3). In general, the estimated maximum power is highest for the farthest alternative parameter and less than 0.80 when closer to the true parameter values. The power plots for large unbalanced dataset are shown in Figure 2. As noticed, the power of the test procedure is good (high) in all four sets.

For both large balanced/unbalanced data, it is observed that the estimated maximum power based on the median is slightly higher than based on the mean and the fluctuation of the estimated power plots based on the mean and median are almost identical.

For small unbalanced data, the estimated power plots are shown in Figure 3. Based on that, the maximum power of the test based on the median estimates is manifested clearly higher than that based on the mean estimates. The pattern of the fluctuation of the power plots based on mean and median are almost similar. However, for small balanced dataset, the maximum power of the test procedure is low (below 0.80) in almost all cases (Refer to Figure 4). The pattern of the fluctuation of the estimated power plots based on mean and median are erratic.

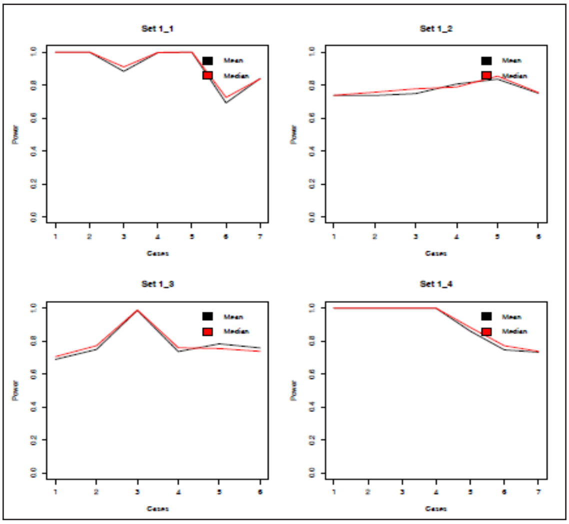

Figure 1. Power Curve with 5% Different Multivariate Characteristics Across Spatial Locations for Large Balanced Dataset

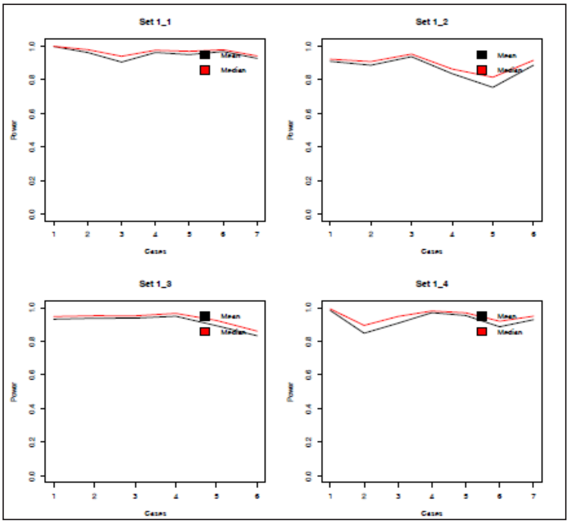

Figure 2. Power Curve with 5% Different Multivariate Characteristics Across Spatial Locations for Large Unbalanced Dataset

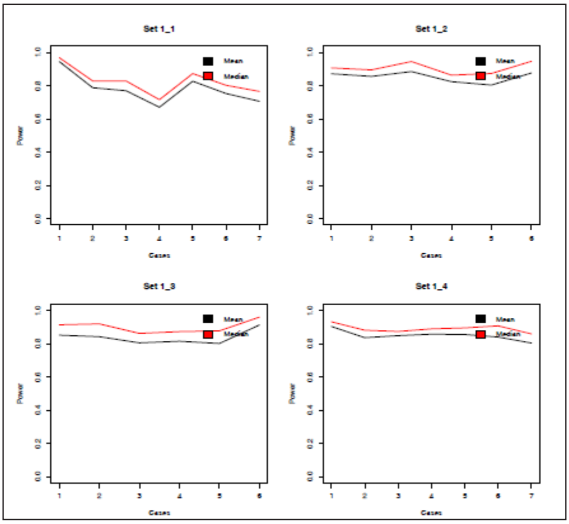

Figure 3. Power Curve with 5% Different Multivariate Characteristics Across Spatial Locations for Small Unbalanced Dataset

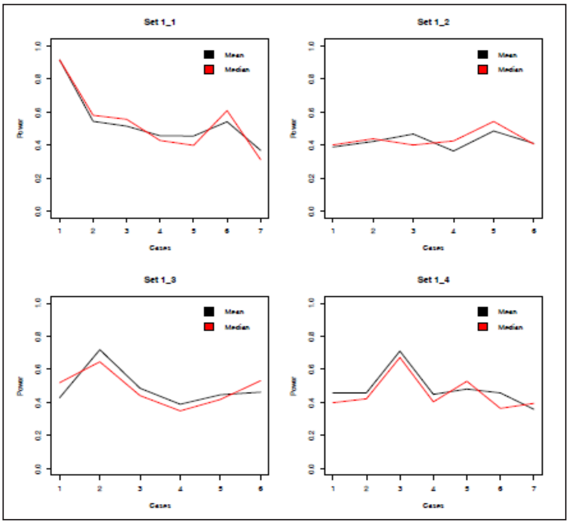

Figure 4. Power Curve with 5% Different Multivariate Characteristics Across Spatial Locations for Small Balanced Dataset

Table 2 presents the results of testing multivariate characteristics across time points. For large balanced data, of the 100 confidence intervals constructed, five (5) failed to contain both the mean and median estimates. For large unbalanced data, of the 50 confidence intervals constructed, one (1) failed to contain the mean estimate and two (2) confidence intervals failed to contain the median estimate. The null hypothesis of constant multivariate characteristics across time points was correctly not rejected with 95% coverage probability for large balanced and/ or unbalanced dataset. On the other hand, for small unbalanced dataset set, of the 35 confidence intervals constructed, two (2) failed to contain the mean estimate and only one (1) interval failed to contain the median estimate. In the case of small balanced dataset, of the 20 confidence intervals constructed, only one (1) confidence interval failed to contain both the mean and median estimates. This can be interpreted that the null hypothesis of constant multivariate characteristics across time point was not also rejected with 95% coverage probability.

Table 2. Multivariate Characteristics Across Time Points for Stationary Bivariate VAR(1) Model for Large and Small Balanced/ Unbalanced Data and 95% Coverage Probability

Sample Size	Criterion	Value	95% Coverage Probability
(N,T) = (100,100)	Mean	0.442	Do not reject H_0 (5)
	Median	0.441	Do not reject H_0 (5)
(N,T) = (70,50)	Mean	0.429	Do not reject H0(1)
	Median	0.415	Do not reject H_0 (2)
(N,T) = (40,35)	Mean	0.445	Do not reject H_0 (2)
	Median	0.439	Do not reject H_0 (1)
(N,T) = (20,20)	Mean	0.463	Do not reject H_0 (1)
	Median	0.447	Do not reject H_0 (1)

Note: Figures in parentheses are the number of normal bootstrap confidence intervals that failed to contain the mean/or median.

The power plots in Figures 5 to 8, show the estimated maximum power with 5% different multivariate characteristics across time points. The estimated maximum power was high in most of the cases in each set of large balanced/ unbalanced data, indicating that the proposed procedure was powerful (see Figures 5 and 6). The pattern of the fluctuation of the estimated power based on mean and median estimates are almost identical with the median estimates which are slightly higher than the mean estimates. However, for small unbalanced/ balanced data, the maximum power of the test procedure was observed to be low in the majority of the cases (refer to Figures 7 and 8).

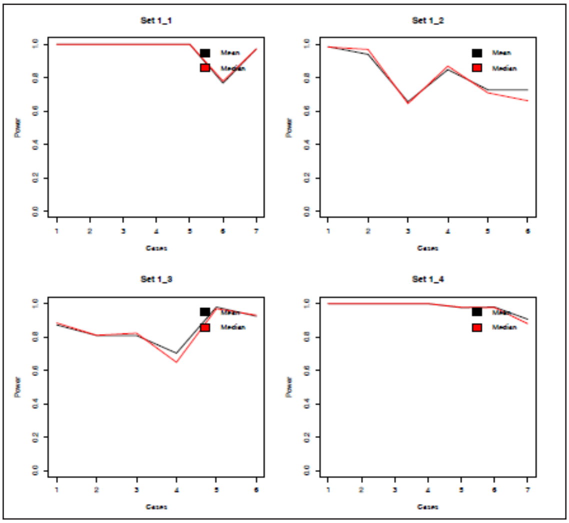

Figure 5. Power Curve with 5% Different Multivariate Characteristics Across Time Points for Large Balanced Dataset

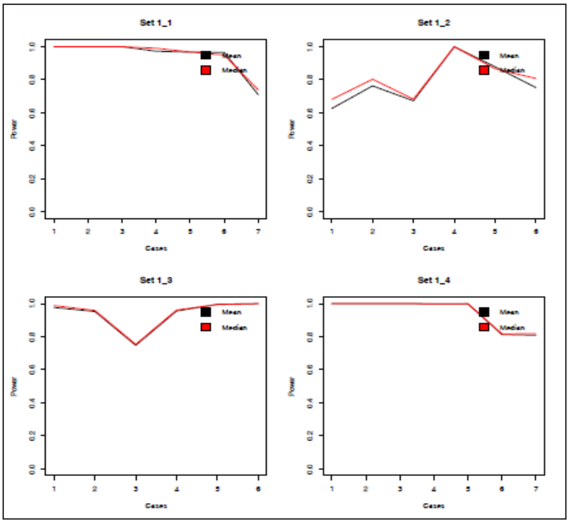

Figure 6. Power Curve with 5% Different Multivariate Characteristics Across Time Points for Large Unbalanced Dataset

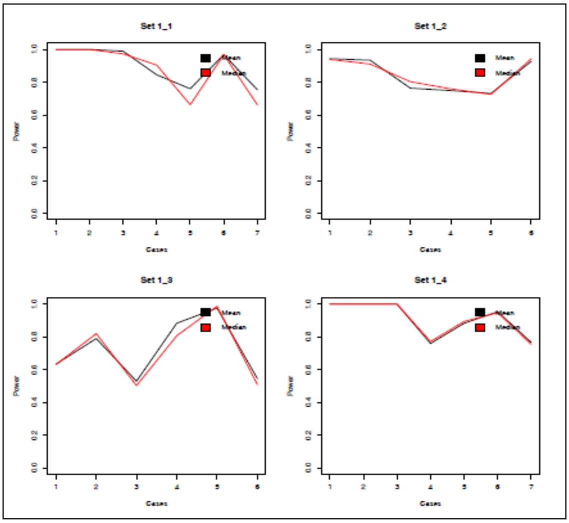

Figure 7. Power Curve with 5% Different Multivariate Characteristics Across Time Points for Small Unbalanced Dataset

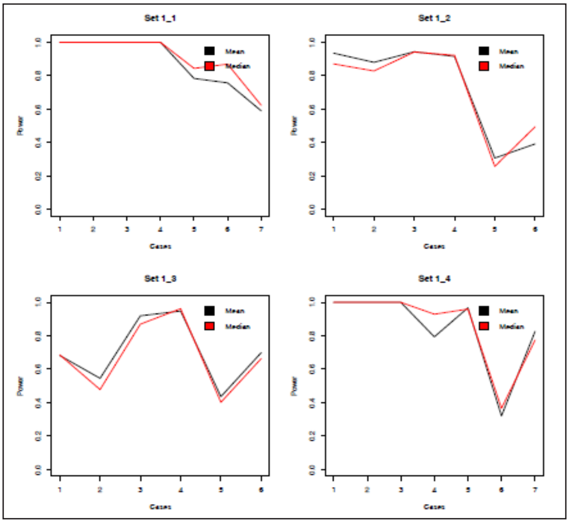

Figure 8. Power Curve with 5% Different Multivariate Characteristics Across Time Points for Small Balanced Dataset

5. Conclusions

The simplest VAR of order 1 with bivariate specification modelling effort is a good foundation for the development of multivariate spatial-temporal models. We have demonstrated the effectiveness of our proposed test procedures on simulated datasets that accurately revealed the true situation. We have also demonstrated that including both spatial and temporal in multivariate model, although difficult, is feasible. We can therefore conclude that the test procedures were able to correctly identify the true situation and are properly sized for large balanced/unbalanced data and are powerful, and so, the tests are robust. However, for small balanced/unbalanced data, the tests were not robust, because of their being less powerful.

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