Analysis of Mother's Day Celebration Via Circular Statistics

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This paper, handles with much emphasis, Mother's Day celebration around the world — a day that is celebrated on various days in different countries. These days are marked in relation to certain historical, religious or mythical events for every nation.

The celebration of mother's day by 152 nations is analyzed using a set of circular statistics procedures to study its characteristics. The frequencies of celebration days are modeled, possible clusters and outliers are defined to assess possible factors that may affect the celebration in a certain date. These factors are found to be culture, language, colonization and neighborhood with insignificant role of religion.

Keywords: Boxplot, cluster, direction, outlier

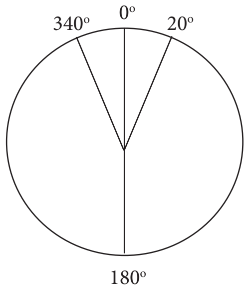
1. Introduction

The analysis of statistical data relies on the nature of its topological distribution. The standard statistical techniques are fit to the linear data, where data are simply presented on a straight line. Another type is circular data, where the circumference of the unit circle offers an appropriate representation. Circular data refer to a set of observations expressed as angles relative to some fixed reference point and distributed within $(0,2\pi)$ radians.

The disparate topologies of the circle and straight line are reflected on the mathematical and statistical treatments of the data. Circle is a closed curve, but not for a line. Moreover, the directions close to the opposite end-points are near neighbor in a circular metric but maximally distant in linear metric.

Applying the conventional linear techniques on circular data may lead to paradoxes. For example, let us consider two angles 20^o and 340^o as illustrated in Figure 1. The arithmetic mean by treating the data as linear observations is 180^o. However, the mean direction of the two directions has to be 0^o. Therefore, special statistical techniques are needed to analyze circular data while taking into account the structure of circular sample space.

Figure 1 Arithmetic and Geometric Mean



The interest in analyzing circular or directional data is as old as the subject of mathematical statistics itself. Circular data are found in many scientific fields where the astronomy was the host soil for the roots of circular data when the Reverend John Mitchell FRS analyzed the angular separation between stars in 1767. The second valuable contribution was in geographical context, where John Playfair in 1902 was the first man who pointed out the requirement of new and different methods to analyze circular data. Fractional part of atomic weights (von Mises, 1918), and source of signals in the case of airplane crashes (Lenth, 1981) are two applications of the circular data in physics.

There are many circular data arising in meteorological studies such as wind and wave directions (Johnson and Wehrly, 1977; Hussin et al., 2004; Gatto and Jammalamadaka, 2007). The number of times a day at which thunderstorms occur and the frequencies of heavy rain in a year (Mardia and Jupp, 2000). Animal navigation (Batschelet, 1981) and spawning times of a particular fish (Lund, 1999a) are two examples of circular data in biology.

Analysis of directions in geological problems also employed circular statistics, as found in the modeling of cross-bedding data (Jones and James, 1969) and the direction of earthquake displacement in terms of the direction steepest decent (Rivest, 1997). Medicine is not far away of circular statistics applications, where the angle of knee flexion as a measure of recovery of orthopedic patients is handled via circular statistics (Jammalamadaka et al., 1986).

Via an extensive survey, we may conclude that social sciences have recently adapted the concepts of the circularity in their data analysis. Few studies were found in this aspect, such as: studies of mental maps to represent the surroundings of respondents (Gordon et al., 1989). Analysis of time patterns in crime incidence (Brunsdon and Corcoran, 2006) and modeling of the casualties in the second Iraq war and suicide cases in Switzerland (Gill and Hangartner, 2010).

Mother's day is celebrated on various days in many parts of the world. These days are marked in relation to certain historical, religious or mythical events for each nation. Given the fact that circular data can be found whenever periodic phenomena occur and that there's not any known published work presenting analysis of mother's day data. Therefore, we perceive mother's day celebration by the 152 nations around the world as (a special day of the year) circular data.

Motivations of this study can be summarized in two main points. First, people from certain nationalities assumed that their nationally celebrated mother's day is a global day. The current issue which will be further explored is for emerging states such as South Sudan—which has recently acclaimed independence—celebrate mother's day. In this regard, we endeavor to examine possible factors defining the tendency of people to celebrate mother's day on a certain given date. This paper aims to achieve the following objectives:

1. To introduce circular statistics to new social issues,
2. To study the characteristics of mother's day data through a statistical approach,
3. To model the frequencies of celebration days,
4. To define possible clusters and outliers of mother's day data,
5. To assess possible factors that may affect the celebration of mother's day on a given date.

The paper starts with an overview of basic circular statistics to provide a consistent framework for readers. Mother's day data are presented and described in Section 3. Modeling, clustering and identifying of possible outliers are explained in Section 4. A discussion on the results of statistical analysis is presented in Section 5.

2. Overview of Circular Statistics

This section reviews some of the descriptive measures for circular data and describes widely used circular distributions which are the circular uniform and von Mises distributions. A holistic discussion approach of circular descriptive measures and probability distributions is available in various monographs (Batschelet, 1981; Fisher, 1993; Mardia and Jupp, 2000; Jammalamadaka and Sen Gupta, 2001).

2.1 Descriptive measures for circular data

Let $\theta_1,...,\theta_N$ be the observations of a random circular variable of size N, the mean direction (preferred direction) is the angle made by the resultant vector $R=(C^2+S^2)^{1/2}$, with the horizontal line, where $C=\sum_{i=1}^{N}\cos\theta_i$ and $S=\sum_{i=1}^{N}\sin\theta_i$. The mean direction is given by

$$\mu_c = \begin{cases} \tan^{-1}(S/C), & \text{if} \quad S \geq 0, C > 0, \\ \dfrac{\pi}{2}, & \text{if} \quad S > 0, C = 0, \\ \tan^{-1}(S/C) + \pi, & \text{if} \quad C < 0, \\ \tan^{-1}(S/C) + 2\pi, & \text{if} \quad S < 0, C \geq 0, \\ \text{undefined}, & \text{if} \quad S = 0, C = 0. \end{cases} \tag{1}$$

The median direction of circular variable defined as an axis (median axis) that divides the circular data into two equal groups (Fisher, 1993). Practically, for any circular data, the median direction is the observation ϕ which minimizes the summation of circular distances, $\mathrm{d}(\varphi) = \pi - \sum_{i=1}^{n} |\pi - |\theta_i - \varphi||$ for $i = 1, \ldots, N$.

Mean resultant length, $\overline{R} = R/N$ is a measure of circular data concentration towards its centre. $\overline{R} \in [0,1]$, when $\overline{R}$ is close to 1, it implies that all directions in the data set are similar. However, $\overline{R} = 0$ does not imply that the directions are spread almost evenly around the circle. For example, any data set of the form $\theta_1, \ldots, \theta_n$, and $\theta_1 + \pi, \ldots, \theta_n + \pi$ has $\overline{R} \approx 0$. The circular variance is defined by the quantity, $V = 1 - \overline{R}$, where $0 \leq V \leq 1$. The smaller values of circular variance refer to a more concentrated data. The quantity $v = (-2\log(1-V)^{½}$ is defined as the sample circular standard deviation with $0 < v < \infty$, where V is the sample circular variance.

2.2 Some circular probability distributions

A circular distribution is a probability distribution whose total probability is concentrated on the circumference of a unit circle and has the following properties: (a) $f(\theta) \geq 0$; (b) $\int_0^{2\pi} f(\theta)\, d\theta = 1$; and (c) $f(\theta) = f(\theta + 2k\pi)$ for any integer k.

If all directions are equally likely, then the probability is uniformly spread out on the circumference of a circle. Thus, the circular uniform distribution with the constant density is given by

$$f(\theta) = \frac{1}{2\pi}, \qquad 0 \leq \theta < 2\pi \tag{2}$$

The other distribution is the von Mises distribution, it was introduced in (von Mises, 1918) to study the deviations of measured atomic weight from integral values. It is the most common distribution considered for unimodal circular variables. The von Mises distribution has been extensively discussed where many inference techniques have been developed. It is denoted by $VM(\mu,\kappa)$, where μ is the mean direction and κ is the concentration parameter. The probability density function for the von Mises distribution is given by

$$f(\theta;\mu,\kappa)=\frac{1}{2\pi\, I_0(\kappa)}\exp\{\kappa\cos(\theta-\mu)\},\ 0\le\theta,\mu<2\pi \text{ and } \kappa\ge 0, \qquad (3)$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero, and it is given by

$$\mathrm{I}_0(\kappa)=\frac{1}{2\pi}\int_0^{2\pi}\exp(\kappa\cos\theta)d\theta=\sum_{r=0}^{\infty}\left(\frac{\kappa}{2}\right)^{2r}\left(\frac{1}{r!}\right)^2. \qquad (4)$$

3. Mother's Day Data

The celebration of mother's day goes back to the ancient civilizations, Egyptians, Romans and Greeks, when societies tended to celebrate goddesses and symbols. Later on mother's day celebrations became associated with different religious, historical or mythical events. Henceforth, it has been celebrated on various days and with different ways in many parts of the world.

The celebration of mother's day differs from one country to another. Some countries consider it a public holiday, to encourage family members to meet. Nowadays, mother's day became one of the most commercially successful occasions in the U.S. where The National Retail Foundation predicted total mother's day spending to reach 14.10 U.S. billion dollars in 2009. Additionally, mother's day became most popular in the U.S. where people largely flock out for dinner according to the National Restaurant Association.

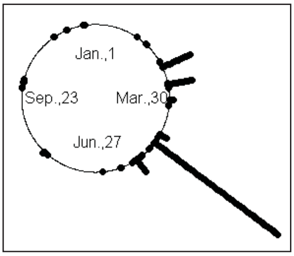
In this study, we consider mother's day celebration dates in 152 nations worldwide in 2011 as published at the free encyclopedia site "Wikipedia." The majority of countries follow the Georgian calendar, while three countries, namely Israel, Iran and Nepal, follow other calendars.

In Israel, mother's day is on the Shevat 30 according to the Hebrew calendar, which corresponds to February 4, 2011. Mother's day in Iran was changed from Azar 25 on the Iranian calendar during the Shah era to be celebrated on Jumada al-Thani 20th in the Islamic calendar (Hijri). The celebration in 2011 corresponds to May 23rd. The Nepalese celebrate the "Mother Pilgrimage fortnight," which falls in the time of dark moon's in the month of Baishak in their calendar, which corresponds to May 3, 2011.

Days of the year range from 1 to 365, day one falls on January 1st and day 365 falls on December 31st. In order to represent a day-of-year in circular form, then 360^o are divided by 365 days. Consequently, each day represents 0.986^o on the circular range. Day one would be then 0.492^o and day 365 would be 359.508^o.

A total of 152 dates are sorted and associated with a day of year orders then the angles are obtained, by this the dates are ready to be treated as circular data. Figure 2 shows circular plot of mother's day data, where dates are distributed on the circumference of circle (i.e. day-of-year). Two long peaks are observed, one at 125.753° which corresponds to May 8th and a shorter peak at 65.589° corresponding to March 8th. These two peaks reveal the bimodality of data.

Figure 2 Circular Plot of Mother's Day Data



A set of descriptive statistics is given in Table 1. The mean direction (preferred date) is at 114.919° (April 27th). Median and mode directions are at 125.753° which corresponds to May 8th. The symmetry of dates distribution can be assumed due to the equality of median and mode directions and the closeness to the mean direction. Although the dates are distributed on the entire days of a year, they are concentrated around the preferred directions.

Table 1 Some Descriptive Measures for Mother's Day Data

Descriptive measure	Value	Date
Mean direction, μ_c	114.919°	27/04/2011
Median, ϕ	125.753°	08/05/2011
Mode	125.753°	08/05/2011
Mean resultant length, $\overline{R}$	0.801	
Variance, V	0.199	
Standard deviation, υ	38.184°	(39 days)
Concentration parameter, κ	2.872	

The mean resultant length is about 0.8 and the concentration parameter is 2.87, which is considered relatively large, (Fisher and Lee, 1992). The variation around the mean direction is represented via the circular standard deviation 38.184° (around 39 days). This leads us to conclude that, dates are highly concentrated around the preferred directions.

Descriptive circular statistics associated with the circular plot shows that mother's day is celebrated throughout the whole year due to the variety of days in which the celebration originated in different countries. It is noteworthy that 79 (53.02%) countries celebrate on the second Sunday of May (May 8, 2011).

4. Analysis of Mother's Day Dates

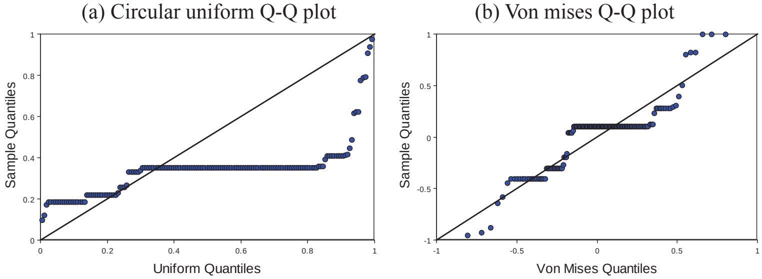
This section investigates possible model of mother's day data using popular circular distributions and the nonparametric kernel density estimates. Cluster analysis and outlier detection procedures are also discussed.

4.1 Modeling Mother's Day data

i- Uniform circular distribution: Circular plot in Figure 2, suggests the rejection of uniformity of mother's day data. Circular uniform Q-Q plot of data is given in Figure 3. (a), it is obvious that the quantiles are far from the diagonal line which means that data are not uniformly distributed.

ii- Von misses distribution: In spite of the two peaks shown in Figure 2, von Misses distribution remains another option to model the mother's day data. Von Mises Q-Q plot is given in Figure 3.(b), although points are not distributed exactly over the diagonal line, they are still closer to be compared to the uniform case. Resultantly, we conclude that data does not follow von Misses distribution.

Figure 3 Q-Q Plots for Mother's Day Data



iii- Nonparametric kernel density estimates: An alternative approach, so-called a nonparametric kernel density estimates (Silverman, 1986) is considered to model the mother's day data. No simple mathematical expression of the density function is given, where the curve is smoothed based on the data. A kernel density estimate function is given by

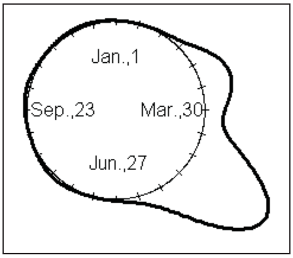
$$\hat{f}(\theta) = (nh)^{-1} \sum_{i=1}^{n} w\left(\frac{\theta - \theta_i}{h}\right) \tag{5}$$

where $w(.)$ usually, but not always, a radically symmetric and unimodal probability density function, such as von Mises distribution. The parameter h is called the smoothing parameter or bandwidth. Problem of choosing the bandwidth is crucial in density estimation. A large bandwidth will over-smooth the density and mask the structure in the data, while a small bandwidth will yield a density estimate that is spiky and very hard to interpret.

Based on the von Mises distribution, $VM(\theta|\theta_i, \kappa)$ the kernel density function with a bandwidth $h = nI_0(\kappa)$ is given by

$$\hat{f}(\theta) = \frac{1}{n\,I_0(\kappa)} \sum_{i=1}^{n} \mathrm{vm}(\theta|\theta_i, \kappa) \tag{6}$$

Figure 4 Kernel Density Curve of Mother's Day Data for $\kappa = 20$



Following the "trial and error approach" to define the bandwidth value, a value of, κ=20 is chosen. Figure 4 shows the kernel density estimates of the mother's day data. It clearly demonstrates the bimodality of data via the two pumps in the first six months of the year.

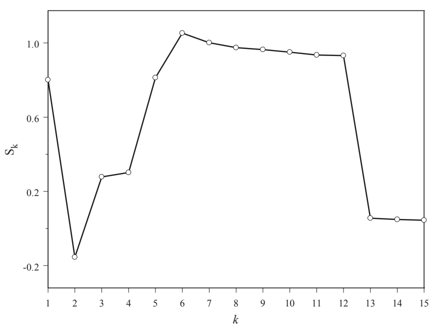
4.2 Cluster analysis of Mother's Day data

Cluster analysis is a vital approach to highlight the possible factors that may gather a set of countries in one group. Cluster analysis in circular data have not received much interest in literature. (Lund, 1999b) proposed a nonparametric statistic for univariate circular data. As in all of the clustering techniques, the circular distance between two circular observations θ_i and θ_j is given by $\delta_{ij} = \pi - |\pi - |\theta_i - \theta_j||$ is proposed to be used as a conventional measure of dissimilarity.

The statistic is derived based on the mean resultant length, where possible clusters are proposed according to the largest arc lengths between observations. The largest k spaces define the k clusters of data. Sequentially, the midpoints of these spaces are used to partition the circle. For k clusters, S_k-statistic measures the concentration of the clusters and is given by $S_k = \sum_{i=1}^{k} (\overline{R}_i - P_i)$, where $\overline{R}_i$ is the mean resultant length of the *i*-th cluster, and P_i is mean resultant length of the *i*-th cluster assuming that the observation within the cluster are uniformly distributed.

The largest value of S_k-statistic indicates the optimal number of clusters. Figure 5 shows a plot of S_k versus *k*, it is noticed that S_2 is a negative, because the data are grouped in two clusters, one cluster constitutes three countries only, namely Thailand (August 12), Costa Rica (August 15), and Antwerp (August 15), with associated angles 220.437°, 223.397° and 223.397° respectively. The P_2 = 0.963 and $\overline{R}_2$ = 0.821 which leads to a negative difference between $\overline{R}_2$ and P_2, equivalently -0.142.

Figure 5 Cluster Statistic for 1 to 20 Clusters



It is obvious that S_k is maximized for k=6, where S_6=1.053, indicating that the distribution of these six clusters are the most significantly concentrated. Table 2 gives the countries for each cluster and the values of $\overline{R}_i$ and P_i for i=1,...,6. .

4.3 Outliers identification in Mother's Day data

The existence of inconsistent observations—which are known as outlier—influences the overall understanding of the characteristics of phenomenon under study. There is a possibility of identifying outliers in circular data despite its close-bounded range. A set of numerical and graphical methods to detect possible outliers in univariate circular data are available (Collect, 1980; Abuzaid et al., 2009, 2011).

Table 3 summarizes the results of two numerical outlier detection procedures, namely, C-statistic and A-statistic. Five countries are identified as outliers based on both detection procedures (Argentina, Belarus, Malawi, Russia and Panama).

Table 2 Mother's Day Data Grouped in Six Possible Clusters

Cluster	Observations	*n*	$\overline{R}_i$	P_i
1	Georgia (Mar. 3). Afghanistan, Albania, Armenia, Azerbaijan, Belarus, Bosnia and Herzegovina, Bulgaria, Kazakhstan, Laos, Macedonia, Moldova, Montenegro, Romania, Russia, Serbia, Ukraine and Vietnam (Mar. 8). Bahrain, Egypt, Jordan, Kuwait, Libya, Lebanon, Oman, Palestine, Saudi Arabia, Sudan, Syria, United Arab Emirates, Yemen, Iraq (Mar. 21). Slovenia (Mar. 25). Ireland, Nigeria, United Kingdom (Apr. 3). Armenia (Apr. 7).	37	0.987	0.961
2	Hungary, Lithuania, Mozambique, Portugal, Spain (May 1). Nepal (May 3). Albania, South Korea, Anguilla, Aruba, Australia, Austria, Bahamas, Bangladesh, Barbados, Belgium, Belize, Bermuda, Bonaire, Brazil, Brunei, Bulgaria, Canada, Chile, China , Colombia, Croatia, Cuba, Curaçao, Cyprus, Czech, Denmark, Dominica, Ecuador, Estonia, Ethiopia, Fiji, Finland, German, Ghana, Greece, Grenada, Honduras, Hong Kong, Iceland, India, Italy, Jamaica, Japan, Latvia, Liechtenstein, Macao, Malaysia, Malta, Myanmar, Netherlands, New Zealand, Pakistan, Papua New Guinea, Peru, Philippines, Puerto Rico, Saint Kitts and Nevis, Saint Lucia, Saint Vincent and the Grenadines, Samoa, Singapore, Sint Maarten, Slovakia, South Africa, Sri Lanka, Suriname, Switzerland, Taiwan, Tanzania, Tonga, Trinidad and Tobago, Turkey, Uganda, Ukraine, United States, Uruguay, Venezuela, Zambia, Zimbabwe (May 8). El Salvador, Guatemala, Mexico (May 10). Paraguay (May 15) Poland (May 26) Bolivia (May, 27) Iran (May 23) Algeria, Domini, France, Haiti, Morocco, Mauritius, Sweden, Tunisia (May 29). Nicaragua (May 30), Mongolia (June 1), Luxembourg (June 12), Kenya (June 26).	104	0.988	0.899
3	Thailand (Aug. 12), Costa Rica, Antwerp (Aug. 15)	3	0.999	0.963
4	Malawi (Oct. 10), Belarus (Oct. 14), Argentina (Oct. 16),	3	0.999	0.963
5	Russia (Nov. 27), Panama (Dec. 8), Indonesia (Dec. 22),	3	0.984	0.943
6	Israel (Feb. 04), Norway (Feb. 13)	2	0.997	0.120

Table 3 Summary of Outliers Detection Procedures

Observation	C-statistic (cut-off point)	A-statistic (cut-off point)
Argentina (Oct.,16)	0.015 (0.020)	0.900 (0.887)
Belarus (Oct.,14)	0.015 (0.014)	0.903 (0.873)
Malawi (Oct.,10)	0.015 (0.014)	0.903 (0.864)
Russia (Nov., 27)	0.014 (0.013)	0.860 (0.831)
Panama (Dec., 8)	0.013 (0.010)	0.818 (0.824)
Indonesia (Dec.,22)	0.012 (0.009)	0.745 (0.805)

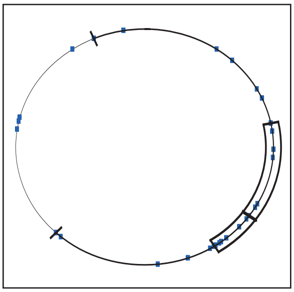
Boxplot is a popular procedure for detecting outliers in univariate linear data. For circular data, a circular version of boxplot with its criterion of the form $v \times$CIQR is developed in (Abuzaid et al., 2011), where CIQR is the circular interquartile range and v is the resistant constant which highly depends on the concentration parameter. It is suggested to use values of v between 1 and 2 when the concentration parameter is between 2 and 3.

Table 4 provides a list of identified outliers for different values of resistant constant, v. Moreover, Figure 6 shows the boxplot of mother's day data for 1.5. Based on the results of C-statistic, A-statistic and circular boxplot, we may conclude that there are four countries that are identified as outliers which are Malawi, Belarus, Argentina, and Russia. Other five countries also have the possibility to be identified as outliers at a higher level of significance, namely Indonesia, Israel, Norway, Thailand, Costa Rica and Antwerp.

Table 4 Summary of the Outliers Detected using Several Values of *v* for Mother's Day Data

v	L_F	U_F	n	Outliers
1.0	Feb. 2	Jun., 262	9	Thailand, Costa Rica, Antwerp, Malawi, Belarus, Argentina, Russia, Panama , Indonesia
1.3	Aug. 15	Dec., 22	5	Malawi, Belarus, Argentina, Russia, Panama
1.5	Aug. 15	Dec., 8	4	Malawi, Belarus, Argentina, Russia
1.7	Aug. 15	Nov., 27	3	Malawi, Belarus, Argentina
2.0	Oct. 14	Nov., 27	1	Argentina

Figure 6 Circular Boxplot of Mother's Day, for *v* =1.5



5. Discussion and Conclusion

Mother's day, as a social occasion, gains special focus from the family members worldwide. This paper has highlighted the fact that nations celebrate mother's day on different dates in a year. A collection of appropriate statistical procedures has been employed to describe the distribution of celebration days. The analysis of data showed that most world nations celebrate it on the second Sunday of May, which falls on May 5, 2011, where other nations marked their celebrations on various other days.

In order to assess the sources of these differences, cluster analysis of circular data was carried out. A total of 152 nations are grouped into six clusters. The largest group contains 104 (68.42%) countries celebrating in May and June, with 79 (51.97%) countries celebrating on May 8th. This group includes the U.S. and a long list of countries with different religions, cultures and history. Other countries belong to this cluster including France and some of its ex-colonies.

The second significant group is composed of 37 (24.34%) countries celebrating in March and April and 17 (11.18%) countries celebrating on March 8th. Some 14 (9.21%) Arab countries celebrate by the beginning of spring (March 21). Only three Arab countries namely Algeria, Morocco, and Tunisia celebrate on May 29 along with France, as they were ex-colonies of France.

The next four groups with a total of 11 (7.23%) countries are mostly identified as outliers using various detection procedures. For instance, in Argentina, the celebration is placed on the third Sunday of October for commercial purposes to reactivate the sales of the second half of October.

We may also conclude that language, neighborhood and colonization play a significant role in clustering the national celebration. On the other hand, religion is an insignificant factor, where Muslim and Christian communities celebrate on varying dates.

South Sudan, which was part of the republic of Sudan, was declared an independent state on September 7, 2011 shares long borders with Sudan, Ethiopia, Uganda, Kenya, Congo and Central Africa. These countries celebrate mother's day in various days, in fact, some countries like Congo and Central Africa have no known celebration dates. This may indicate that South Sudan may celebrate mother's day on the second Sunday of May just like its neighbor Uganda and the majority of the Nations since English is the official state language. Another less likely alternative is to celebrate on the first day of the spring (March 21st) as in Sudan and most Arab countries."

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